Reconstruction of Black Hole Metric Perturbations from Weyl Curvature

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Perturbation theory of rotating black holes is usually described in terms of Weyl scalars ψ_4 and ψ_0, which each satisfy Teukolsky's complex master wave equation and respectively represent outgoing and ingoing radiation. On the other hand metric perturbations of a Kerr hole can be described in terms of (Hertz-like) potentials Ψ in outgoing or ingoing *radiation gauges*. In this paper we relate these potentials to what one actually computes in perturbation theory, i.e ψ_4 and ψ_0. We explicitly construct these relations in the nonrotating limit, preparatory to devising a corresponding approach for building up the perturbed spacetime of a rotating black hole. We discuss the application of our procedure to second order perturbation theory and to the study of radiation reaction effects for a particle orbiting a massive black hole.

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I. INTRODUCTION

The spherically symmetry of a Schwarzschild black hole background allows for a multipole decomposition of metric perturbations, even in the time domain. These were studied originally by Regge and Wheeler [1] for odd-parity perturbations and by Zerilli [2] for the even-parity case. Moncrief [3] has given a gauge-invariant formulation of the problem, in terms of the three-metric perturbations. The two degrees of freedom of the gravitational field are described in terms of two waveforms, $\phi^\pm$ satisfying a simple wave equation

$$-\frac{\partial^2\phi^\pm_{(\ell m)}}{\partial t^2}+\frac{\partial^2\phi^\pm_{(\ell m)}}{\partial r*^2}-V_{\ell^\pm}(r)\phi^\pm_{(\ell m)}=0\,. \tag{1.1}$$

Here $r^*\equiv r+2M\ln(r/2M-1)$, and $V_{\ell^\pm}(r)$ are the Zerilli and Regge–Wheeler potentials respectively.

Given the solution to the wave equation (1.1) one can reconstruct explicitly both the even and odd parity metric perturbations of a Schwarzschild background in the Regge–Wheeler gauge [4, 5]. This permits a complete description of the perturbative spacetime. But the Regge–Wheeler gauge is unfortunately not asymptotically flat, and in order to extract radiation information for a second order perturbative expansion one has to perform a new gauge transformation [6]. Moreover, the desirable properties of the the Regge–Wheeler gauge being unique and invertible are lost in the case the background is a *rotating* black hole, i.e. represented by the Kerr metric, where no effective multipole decomposition is yet known.

There is an independent formulation of the perturbation problem derived from the Newman-Penrose formalism [7] that is valid for perturbations of rotating black holes.[8] This formulation fully exploits the null structure of black holes to decouple the curvature perturbation equations into a single wave equation that, in Boyer-Lindquist coordinates (t,r,θ,φ), can be written as:

$$\begin{aligned}&\Bigg\{\left[a^2\sin^2\theta-\frac{(r^2+a^2)^2}{\Delta}\right]\partial_{tt}-\frac{4Mar}{\Delta}\partial_{t\varphi}\\
&-2s\left[(r+ia\cos\theta)-\frac{M(r^2-a^2)}{\Delta}\right]\partial_t\\
&+\Delta^{-s}\partial_r\left(\Delta^{s+1}\partial_r\right)+\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\right)\\
&+\left(\frac{1}{\sin^2\theta}-\frac{a^2}{\Delta}\right)\partial_{\varphi\varphi}+2s\left[\frac{a(r-M)}{\Delta}+\frac{i\cos\theta}{\sin^2\theta}\right]\partial_\varphi\\
&-\left(s^2\cot^2\theta-s\right)\Bigg\}\psi=4\pi\Sigma T\,,\end{aligned} \tag{1.2}$$

where M is the mass of the black hole, a its angular momentum per unit mass, $\Sigma\equiv r^2+a^2\cos^2\theta$, and $\Delta\equiv r^2-2Mr+a^2$. The source term T is built up from the energy-momentum tensor [8]. Gravitational perturbations, corresponding to $s=\pm2$, are compactly described in terms of contractions of the Weyl tensor with a null tetrad. The components of the tetrad (also given in Ref. [8]) are appropriately chosen along the repeated principal null directions of the background spacetime [see Eq. (1.4) below]. The resulting gauge and (infinitesimally) tetrad invariant components of the Weyl curvature are given by

$$\psi=\begin{cases}\rho^{-4}\psi_4\equiv-\rho^{-4}C_{n\bar{m}n\bar{m}} & \text{for}\quad s=-2\\ \psi_0\equiv-C_{lmlm} & \text{for}\quad s=+2\end{cases}\,, \tag{1.3}$$

where an overbar means complex conjugation and ρ is given in Eq. (1.5) below. Asymptotically, the leading behavior of the field ψ represents either the outgoing radiative part of the perturbed Weyl tensor, $(s=-2)$, or the ingoing radiative part, $(s=+2)$.

The components of the Boyer–Lindquist null tetrad for

the Kerr background are given by

$$(l^\alpha) = \left(\frac{r^2+a^2}{\Delta}, 1, 0, \frac{a}{\Delta}\right), \quad (1.4a)$$

$$(n^\alpha) = \frac{1}{2\left(r^2+a^2\cos^2\theta\right)}\left(r^2+a^2, -\Delta, 0, a\right), \quad (1.4b)$$

$$(m^\alpha) = \frac{1}{\sqrt{2}(r+ia\cos\theta)}\left(ia\sin\theta, 0, 1, i/\sin\theta\right). \quad (1.4c)$$

With this choice of the tetrad the non-vanishing spin coefficients are

$$\begin{aligned}
\rho &= -\frac{1}{(r-ia\cos\theta)}, \quad \beta = -\bar{\rho}\frac{\cot\theta}{2\sqrt{2}}, \\
\pi &= ia\rho^2\frac{\sin\theta}{\sqrt{2}}, \quad \tau = -ia\rho\bar{\rho}\frac{\sin\theta}{\sqrt{2}}, \\
\mu &= \rho^2\bar{\rho}\,\frac{\Delta}{2}, \quad \alpha = \pi - \bar{\beta}, \\
\gamma &= \mu + \rho\bar{\rho}\frac{(r-M)}{2}, \qquad (1.5)
\end{aligned}$$

and the only non-vanishing Weyl scalar in the background is

$$\psi_2 = M\rho^3. \quad (1.6)$$

Analogously to the Zerilli-Regge-Wheeler waveforms, ψ_4 can be directly used to compute energy and momentum radiated at infinity, but it remains to relate it to metric perturbations. Chandrasekhar [9] studied a way to to obtain metric perturbations of the Kerr metric, but it was proved by Price and Ipser [10] that this choice is not a proper gauge, namely is an incomplete constraint on the coordinates.

Chrzanowski [11] generalized work of Cohen and Kegeles [12] on Hertz potentials to the gravitational perturbations of the Kerr metric. In Ref. [11] explicit expressions are given for *homogeneous* metric perturbations in the frequency domain. Wald [13] subsequently showed that the expressions given in Ref. [11] do not lead to *real* metric perturbations. Cohen and Kegeles [14] then corrected their expressions and gave explicit equations (see Sec. II) relating metric perturbations to a gravitational Hertz potential, Ψ that fulfills Eq. (1.2), but that is different from ψ_4 or ψ_0. In those works no explicit method was given for determining Ψ in any specific astrophysical problem.

In Sec. III we provide the explicit formulae relating a gravitational Hertz potential to ψ_4 or ψ_0 in the time domain, hence to the given initial data defining the astrophysical model one wants to evolve (See Ref. [15] for the 3+1 decomposition of the Weyl scalars). These allow one to define the *outgoing* and *ingoing* radiation gauges that are asymptotically flat at future infinity and regular on the event horizon respectively. Such gauges have been found [16] especially well suited for computing second order perturbations of a Kerr hole and, once generalized in presence of matter, can provide a first step toward computing radiation reaction (self-force) corrections [4] to the trajectory of a particle orbiting a rotating black hole.

Vacuum second order perturbations have a direct application to the close limit expansion of the final merger stage of binary black holes with comparable masses. Perturbation theory in conjunction with limited full numerical simulations has proved to be an extremely powerful tool to compute waveforms from binary black holes from near the innermost stable circular orbit [17, 18]. The combination of radiation reaction correction plus second order perturbations provides a formidable tool for computing gravitational radiation from binary black hole–neutron star systems, and is particularly relevant to the computation of template banks for ground based interferometers such as LIGO/VIRGO/GEO about to enter on-line, as well as space missions such as LISA, sensitive to supermassive black hole binary systems. Thus in order to incorporate these improvements to our theoretical predictions it is imperative to know how to build up explicit metric perturbations around a Kerr background.

II. FORMULATION OF THE PROBLEM

We use the notation

$$g_{\mu\nu} = g_{\mu\nu}^{Kerr} + h_{\mu\nu} \quad (2.1)$$

to describe metric perturbations.

A. ingoing and outgoing radiation gauges

Chrzanowski [11], and Cohen and Kegeles [14] found two convenient gauges that allow one to invert the metric perturbations in terms of potentials Ψ_{IRG} or Ψ_{ORG} satisfying the same wave equations as the Weyl scalars $\rho^{-4}\psi_4$ or ψ_0 respectively.

In the *ingoing radiation* gauge (IRG) we have

$$\begin{aligned}
h_{ll} &= h_{\mu\nu}l^\mu l^\nu = 0; \quad h_{ln} = h_{\mu\nu}l^\mu n^\nu = 0, \\
h_{lm} &= h_{\mu\nu}l^\mu m^\nu = 0; \quad h_{l\overline{m}} = h_{\mu\nu}l^\mu \overline{m}^\nu = 0, \\
h_{m\overline{m}} &= h_{\mu\nu}m^\mu \overline{m}^\nu = 0, \qquad (2.2)
\end{aligned}$$

and the homogeneous (for vacuum) metric components can be written, in the time domain, in terms of solutions to the wave equation for $\rho^{-4}\psi_4$ only, as follows:[27]

$$\begin{aligned}
h_{nn}^{IRG} &= -\left\{(\delta+\overline{\alpha}+3\beta-\tau)(\delta+4\beta+3\tau)\right\} \\
&\quad (\Psi_{IRG}) + c.c. \qquad (2.3a) \\
h_{\overline{m}\overline{m}}^{IRG} &= -\left\{(D-\rho)(D+3\rho)\right\}(\Psi_{IRG}), \qquad (2.3b) \\
h_{(n\overline{m})}^{IRG} &= -\frac{1}{2}\left\{(\delta-\overline{\alpha}+3\beta-\overline{\pi}-\tau)(D+3\rho)+\right. \\
&\quad \left.(D+\overline{\rho}-\rho)(\delta+4\beta+3\tau)\right\}(\Psi_{IRG}), \quad (2.3c)
\end{aligned}$$

where *c.c.* stands for the complex conjugate part of the whole object to ensure that the metric be real [13, 14]. Note that in this gauge the metric potential has the properties of being transverse ($h_{\mu\nu}l^\mu = 0$) and traceless ($h_\mu^\mu = 0$) at the future horizon and past null infinity. It

is thus a suitable gauge to study gravitational radiation effects near the event horizon.

The complementary (adjoint) gauge to the ingoing radiation gauge is the *outgoing radiation* gauge (ORG), which can be obtained from the IRG upon exchange of the tetrad vectors $l \leftrightarrow n$, $\overline{m} \leftrightarrow m$ and the appropriate renormalization. It satisfies:

$$h_{nn} = 0 = h_{ln} = 0 = h_{nm} = 0 = h_{n\overline{m}} = 0 = h_{m\overline{m}}. \tag{2.4}$$

The metric potential has now the property of being transverse ($h_{\mu\nu}n^{\mu} = 0$) and traceless ($h^{\mu}_{\mu} = 0$) at the past horizon and future null infinity. It is thus an example of a suitable asymptotically flat gauge in which to directly compute radiated energy and momenta at infinity.

In this gauge, the homogeneous metric components can be written in terms of solutions to the wave equation for ψ_0, as

$$\begin{aligned} h_{ll}^{ORG} &= -\rho^{-4}\left\{(\overline{\delta} - 3\alpha - \overline{\beta} + 5\pi)(\overline{\delta} - 4\alpha + \pi)\right\} \\ &\quad (\Psi_{ORG}) + c.c. & \text{(2.5a)} \\ h_{mm}^{ORG} &= -\rho^{-4}\left\{(\widehat{\Delta} + 5\mu - 3\gamma + \overline{\gamma})(\widehat{\Delta} + \mu - 4\gamma)\right\} \\ &\quad (\Psi_{ORG}), & \text{(2.5b)} \\ h_{(lm)}^{ORG} &= -\frac{1}{2}\rho^{-4}\Big\{(\overline{\delta} - 3\alpha + \overline{\beta} + 5\pi + \overline{\tau})(\widehat{\Delta} + \mu - 4\gamma) \\ &\quad +(\widehat{\Delta} + 5\mu - \overline{\mu} - 3\gamma - \overline{\gamma})(\overline{\delta} - 4\alpha + \pi)\Big\} \\ &\quad (\Psi_{ORG}), & \text{(2.5c)} \end{aligned}$$

where the directional derivatives are $D = l^{\mu}\partial_{\mu}, \widehat{\Delta} = n^{\mu}\partial_{\mu}, \delta = m^{\mu}\partial_{\mu}$, and the rest of the Greek letters represent the usual notation for spin coefficients.[7]

B. 4th order equations for the potential

From the evolution of the Teukolsky equation we can obtain ψ_4 and ψ_0, the two gauge and tetrad invariant objects representing outgoing and ingoing radiation respectively. To relate the unknown potential Ψ to them we use their definitions (1.3) to obtain

$$\begin{aligned} \psi_0 &= DDDD\left[\bar{\Psi}_{IRG}\right], \quad \text{and} & \text{(2.6)} \\ \rho^{-4}\psi_4 &= \frac{1}{4}\left[\bar{\mathcal{L}}\bar{\mathcal{L}}\bar{\mathcal{L}}\bar{\mathcal{L}}\bar{\Psi}_{IRG} - 12\rho^{-3}\psi_2\partial_t\Psi_{IRG}\right], & \text{(2.7)} \end{aligned}$$

where $\bar{\mathcal{L}} = \bar{\eth} + ia\sin\theta\partial_t$ and $\bar{\eth} = -[\partial_\theta + s\cot\theta - i\csc\theta\partial_\varphi]$, while for the outgoing radiation gauge we have

$$\begin{aligned} \rho^{-4}\psi_4 &= \Delta^2\widehat{\Delta}\widehat{\Delta}\widehat{\Delta}\widehat{\Delta}\left[\Delta^2\bar{\Psi}_{ORG}\right], \quad \text{and} & \text{(2.8)} \\ \psi_0 &= \frac{1}{4}\left[\mathcal{L}\mathcal{L}\mathcal{L}\mathcal{L}\bar{\Psi}_{ORG} + 12\rho^{-3}\psi_2\partial_t\Psi_{ORG}\right]. & \text{(2.9)} \end{aligned}$$

In order to obtain an expression for the potentials in terms of the known quantities ψ_4 or ψ_0, one has to invert a fourth order differential equation where ψ_4 or ψ_0 act as source terms. This will be the central task of our paper.

III. EXPLICIT SOLUTION FOR NONROTATING BLACK HOLES

A. Master equation

The key observation here is that for $a = 0$ the differential operator $\mathcal{L}$ acting on the potentials in Eqs. (2.7) and Eqs. (2.9) contains only angular derivatives. Since for the spherically symmetric background we can decompose the potentials into spherical harmonics of spin weight s and, from [19], we have

$$\begin{aligned} {}_s\bar{Y}_{\ell m} &= (-)^{m+s}{}_{-s}Y_{\ell,-m}, & \text{(3.1a)} \\ \bar{\eth}\bar{\eth}\bar{\eth}\bar{\eth}\left[{}_{-2}\bar{Y}_{\ell m}\right] &= (-)^{m-2}\frac{(\ell+2)!}{(\ell-2)!}\,{}_{-2}Y_{\ell,-m}, & \text{(3.1b)} \\ \eth\eth\eth\eth\left[{}_{2}\bar{Y}_{\ell m}\right] &= (-)^{m+2}\frac{(\ell+2)!}{(\ell-2)!}\,{}_{2}Y_{\ell,-m}, & \text{(3.1c)} \end{aligned}$$

we obtain a first order relationship among ψ_4 or ψ_0 and the IRG or ORG potentials decomposed into multipoles

$$\begin{aligned} \left[\rho^{-4}\psi_4\right]^{\pm} &= \pm\frac{(\ell+2)!}{4(\ell-2)!}\Psi^{\pm}_{IRG} - 3M\partial_t\Psi^{\pm}_{IRG}, & \text{(3.2)} \\ \psi_0^{\pm} &= \pm\frac{(\ell+2)!}{4(\ell-2)!}\Psi^{\pm}_{ORG} + 3M\partial_t\Psi^{\pm}_{ORG}, & \text{(3.3)} \end{aligned}$$

where we have used the notation

$$\psi^{\pm} = \frac{1}{2}\left[\,\psi^{\ell,m} \pm (-)^m\overline{\psi^{\ell,-m}}\,\right] \tag{3.4}$$

for all fields decomposed into multipoles.

Since Ψ_{IRG} and Ψ_{ORG} satisfy the master equation (1.2) for spin $s = \mp 2$ respectively we can eliminate from this equation all time derivatives by replacing Eqs. (3.2) or (3.3) and its time derivatives into the Teukolsky equation. This leads to the following equation for the IRG / ORG potentials, both represented here by Ψ

$$\begin{aligned} &\Delta^{-s}\partial_r[\Delta^{s+1}\partial_r\Psi^{\pm}] - \frac{r^4}{\Delta}(\Omega_{AS})^2\Psi^{\pm} \\ &\pm 2sr(\Omega_{AS})(\frac{Mr}{\Delta} - 1)\Psi^{\pm} \\ &-(\ell - s)(\ell + s + 1)\Psi^{\pm} = F^{\pm} \end{aligned} \tag{3.5}$$

where

$$\Psi = \begin{cases} \Psi_{IRG} & \text{for} \quad s = -2 \\ \Psi_{ORG} & \text{for} \quad s = +2 \end{cases}, \tag{3.6}$$

$$\Omega_{AS} = \frac{1}{12M}\frac{(\ell+2)!}{(\ell-2)!}, \tag{3.7}$$

and the source term is

$$\begin{aligned} F^{\pm} \quad &:= -\frac{r^4}{3M\Delta}(\partial_t\psi^{\pm}) \mp \frac{r^4}{3M\Delta}(\Omega_{AS})(\psi^{\pm}) \\ &+2s\frac{r}{3M}(Mr/\Delta - 1)(\psi^{\pm}). \end{aligned} \tag{3.8}$$

The equation (3.5) is our fundamental equation to solve for the potential in terms of the known fields ψ_4 or ψ_0 that appear in the source terms. The key observations here are to note that, separately for the "plus" and "minus" parts: i) the left hand side of this equation contains the Teukolsky operator in the *frequency* domain for the algebraically special frequencies, $\omega = \pm i\Omega_{AS}$, and ii) the source terms in (3.8) are the Cauchy data for the Teukolsky operator in terms of ψ_4 or ψ_0, precisely as they would appear in a Laplace transform approach to Eq. (1.2) [See Eqs. (A2-A3) of Ref. [20]]. Notably, we have arrived to this equation working in the time domain, *without* any frequency decomposition.

B. solution

The single frequency appearing on the right hand side of Eq. (1.2) is precisely an algebraic special frequency. Algebraically special perturbations of black holes excite gravitational waves which are either purely ingoing or outgoing. Hence only one of ψ_4 *or* ψ_0 is non-zero while the other vanishes. Chandrasekhar [21] has obtained the explicit form of the algebraic special perturbations of the Kerr black hole. This is a remarkable fact, because we can then use these analytic solutions to the Teukolsky equation for algebraic special perturbations to construct explicit solutions to (3.5).

For Schwarzschild, Chandrasekhar [21] gives two algebraic special solutions

$$y_1(r) = \left[1 + \frac{\lambda r}{M} + \frac{\lambda^2 r^2}{3M^2} + \frac{\lambda^2(\lambda+1)r^3}{9M^3}\right] e^{-\Omega_{AS} r^*}, \quad (3.9)$$

$$y_2(r) = \left[1 - \frac{(\lambda+1)r}{3M}\right] r^2 e^{+\Omega_{AS} r^*}, \quad (3.10)$$

where $\lambda = (\ell-1)(\ell+2)/2$. Note that y_2 gives rise to an algebraically special solution for ψ_4, $(s=-2)$, at frequency $\omega^-_{AS} = -i\Omega_{AS}$ and for $\Delta^2\psi_0$, $(s=+2)$, at frequency $\omega^+_{AS} = +i\Omega_{AS}$, while y_1 gives rise to an algebraically special solution for ψ_4 at frequency ω^+_{AS} and for $\Delta^2\psi_0$ at frequency ω^-_{AS}.

These two solutions satisfy the "plus" and "minus" parts of Eq. (3.5) for $\Psi^{\pm}_{IRG}$ $(s=+2)$ and vice-versa for $\Psi^{\mp}_{ORG}$ $(s=-2)$. A second set of independent solutions can be found:

$$z_1(r) = y_1(r)\int_{2M}^{r} \frac{W(r')}{y_1(r')^2}\, dr', \quad (3.11)$$

$$z_2(r) = y_2(r)\int_{\infty}^{r} \frac{W(r')}{y_2(r')^2}\, dr', \quad (3.12)$$

where the Wronskian of the solutions is

$$\begin{aligned} W(r) :&= y_{1,2}(r)\partial_r z_{1,2}(r) - z_{1,2}(r)\partial_r y_{1,2}(r) \\ &= \Delta(r) = r(r-2M). \end{aligned} \quad (3.13)$$

Note: z_1 and z_2 are not algebraically special solutions, although they are each a homogeneous solution to their respective Teukolsky equation at an algebraically special frequency.

Making use of these solutions, the explicitly expression for the potential can be written as follows:

$$\begin{aligned} \Psi^+_{IRG}(r) &= -y_1(r)\int_{2M}^{r} \frac{z_1(r')F^+(r')}{\Delta(r')^2}\, dr' \\ &- z_1(r)\int_{r}^{\infty} \frac{y_1(r')F^+(r')}{\Delta(r')^2}\, dr', \end{aligned} \quad (3.14)$$

$$\begin{aligned} \Psi^-_{IRG}(r) &= z_2(r)\int_{2M}^{r} \frac{y_2(r')F^-(r')}{\Delta(r')^2}\, dr' \\ &+ y_2(r)\int_{r}^{\infty} \frac{z_2(r')F^-(r')}{\Delta(r')^2}\, dr', \end{aligned} \quad (3.15)$$

valid for each hypersurface where ψ_0 or ψ_4, entering in F given by Eq. (3.5), are evaluated.

The equations for the ORG are obtained by exchanging "plus" and "minus" parts above (cf. Eq. (A3) and (A4)), and by adopting the corresponding dependence of F on the components of ψ [See Eq. (1.3)].

IV. APPLICATIONS

A. Metric perturbations (IRG)

Using the Regge-Wheeler notation [1] for metric perturbations, conditions (2.2) read

$$\begin{aligned} &h_0^\ell(r,t)^{(\mathrm{even,odd})} = -(1-2M/r)\, h_1^\ell(r,t)^{(\mathrm{even,odd})}, \\ &G^\ell(r,t) = \frac{2}{\ell(\ell+1)} K^\ell(r,t), \\ &H_0^\ell(r,t) = H_2^\ell(r,t) = -H_1^\ell(r,t). \end{aligned} \quad (4.1)$$

We can now explicitly compute the metric perturbations (2.3) or (2.5) in terms of the computed ψ_0 or ψ_4:[28]

$$\begin{aligned} &[h_{(n\overline{m})}]^\ell \\ &= \left\{h_0^\ell(r,t)^{(\mathrm{even})} - i h_0^\ell(r,t)^{(\mathrm{odd})}\right\} \frac{\sqrt{\ell(\ell+1)}}{\sqrt{2}r}\ {}_{-1}Y_{\ell m} \\ &= \left\{\left(\frac{y_1'}{r y_1} + \frac{C_\ell}{12M}\frac{1}{(1-2M/r)} - \frac{2}{r^2}\right)[\Psi^+_{IRG}] \right. \\ &\quad -\frac{\Delta}{r y_1}\int_r^\infty \frac{y_1}{\Delta^2}[F^+]\, dr' - \frac{[\rho^{-4}\psi_4^+]}{3M(r-2M)} \\ &\quad + \left(\frac{y_2'}{r y_2} - \frac{C_\ell}{12M}\frac{1}{(1-2M/r)} - \frac{2}{r^2}\right)[\Psi^-_{IRG}] \\ &\quad \left. + \frac{\Delta}{r y_2}\int_{2M}^r \frac{y_2}{\Delta^2}[F^-]\, dr' - \frac{[\rho^{-4}\psi_4^-]}{3M(r-2M)}\right\} \\ &\quad \times\sqrt{2(\ell+2)(\ell-1)}\ {}_{-1}Y_{\ell m}, \end{aligned} \quad (4.2)$$

where $C_\ell = (l+2)!/(l-2)!$, and

$$\begin{aligned} [h_{nn}]^\ell &= (1-2M/r)\ H_0^\ell(r,t)\ {}_0Y_{\ell m} \\ &= \frac{\rho^2}{2}[\Psi^+_{IRG}]\sqrt{\frac{(\ell+2)!}{(\ell-2)!}}\ {}_0Y_{\ell m}, \end{aligned} \quad (4.3)$$

$$
\begin{aligned}
[h_{\overline{mm}}]^\ell &= \frac{1}{2}\left\{G(r,t)^\ell + i\frac{h_2(r,t)^\ell}{r^2}\right\}\sqrt{\frac{(\ell+2)!}{(\ell-2)!}}\ {}_{-2}Y_{\ell m} \\
&= [h_{\overline{mm}}]^{(\mathrm{even})} + [h_{\overline{mm}}]^{(\mathrm{odd})} \qquad (4.4)
\end{aligned}
$$

where

$$
\begin{aligned}
[h_{\overline{mm}}]^{(\mathrm{even})} = \Bigg\{ & 1/6\,\frac{[\Psi^+_{IRG}]\left(12\,M^2+r^2C_l\right)y_1'}{y_1 r\,(r-2\,M)\,M} \\
& -1/6\,\frac{(\Delta)\left(12\,M^2+r^2C_l\right)}{y_1 r\,(r-2\,M)\,M}\int_r^\infty \frac{y_1}{\Delta^2}[F^+]\ dr' \\
& -2/3\,\frac{r^2[\rho^{-4}\partial_t\psi_4^+]}{(r-2\,M)^2\,M} - 2/3\,\frac{r\frac{\partial}{\partial r}[\rho^{-4}\psi_4^+]}{(r-2\,M)\,M} \\
& -1/18\,\frac{[\rho^{-4}\psi_4^+]\left(-36\,r+84\,M+r^2C_l\right)}{(r-2\,M)^2\,M} \\
& +\frac{1}{72}\,\Big(r^3{C_l}^2+12\,r\left(6\,l^2+6\,l+7\,C_l-12\right)M^2 \\
& -144\,(l+2)\,(l-1)\,M^3-36\,r^2C_lM\Big) \\
& \times\frac{[\Psi^+_{IRG}]}{(r-2\,M)^2\,M^2r}\Bigg\}\ {}_{-2}Y_{\ell m}, \qquad (4.5)
\end{aligned}
$$

and

$$
\begin{aligned}
[h_{\overline{mm}}]^{(\mathrm{odd})} = \Bigg\{ & 1/6\,\frac{[\Psi^-_{IRG}]\left(12\,M^2-r^2C_l\right)y_2'}{y_2 r\,(r-2\,M)\,M} \\
& +1/6\,\frac{(\Delta)\left(12\,M^2-r^2C_l\right)}{y_2 r\,(r-2\,M)\,M}\int_{2M}^r \frac{y_2}{\Delta^2}[F^-]\ dr' \\
& -2/3\,\frac{r^2[\rho^{-4}\partial_t\psi_4^-]}{(r-2\,M)^2\,M} - 2/3\,\frac{r\frac{\partial}{\partial r}[\rho^{-4}\psi_4^-]}{(r-2\,M)\,M} \\
& -1/18\,\frac{[\rho^{-4}\psi_4^-]\left(-36\,r+84\,M-r^2C_l\right)}{(r-2\,M)^2\,M} \\
& +\frac{1}{72}\,\Big(r^3{C_l}^2+12\,r\left(6\,l^2+6\,l-7\,C_l-12\right)M^2 \\
& -144\,(l+2)\,(l-1)\,M^3+36\,r^2C_lM\Big) \\
& \times\frac{[\Psi^-_{IRG}]}{(r-2\,M)^2\,M^2r}\Bigg\}\ {}_{-2}Y_{\ell m}. \qquad (4.6)
\end{aligned}
$$

In writing these we have explicitly lowered the order of the derivatives of the potential by making use of the following identities

$$
\begin{aligned}
&\partial_r\left([\Psi^+]\right) = \\
&\left(\frac{\partial_r y_1}{y_1}\right)[\Psi^+] - \frac{\Delta}{y_1}\int_r^\infty \left(\frac{y_1}{\Delta^2}\right)[F^+]\ dr', \qquad (4.7)
\end{aligned}
$$

$$
\begin{aligned}
&\partial_r\left([\Psi^-]\right) = \\
&\left(\frac{\partial_r y_2}{y_2}\right)[\Psi^-] + \frac{\Delta}{y_2}\int_{2M}^r \left(\frac{y_2}{\Delta^2}\right)[F^-]\ dr', \qquad (4.8)
\end{aligned}
$$

and directly from Eq. (3.2)

$$
\partial_t\Psi^\pm_{IRG} = \pm\frac{(\ell+2)!}{12M(\ell-2)!}\Psi^\pm_{IRG} - \frac{1}{3M\rho^4}\psi_4^\pm. \qquad (4.9)
$$

B. Metric perturbations (ORG)

Using the Regge-Wheeler notation [1] for metric perturbations conditions (2.4) read

$$
\begin{aligned}
& h_0^\ell(r,t)^{(\mathrm{even,odd})} = (1-2M/r)\,h_1^\ell(r,t)^{(\mathrm{even,odd})}, \\
& G^\ell(r,t) = \frac{2}{\ell(\ell+1)}K^\ell(r,t), \\
& H_0^\ell(r,t) = H_2^\ell(r,t) = H_1^\ell(r,t). \qquad (4.10)
\end{aligned}
$$

The explicit metric perturbations can be found directly from the previous subsection, Eqs. (4.2)-(4.6) upon exchanges of the tetrad contractions $l \to n$ and $m \to \bar{m}$, and the consequent change in normalizations, s, and potentials, as described throughout the paper.

V. DISCUSSION

We have explicitly computed the metric perturbations of a nonrotating black hole in terms of the Weyl scalars ψ_4 and ψ_0 which can be computed directly by solving the Teukolsky equation for any appropriate astrophysical scenario, given the corresponding initial data. In doing so we had to invert Eqs. (2.7) or (2.9). This was performed making explicit use of the multipole decomposition of the potential, Weyl scalars and metric. The extension of this procedure to the rotating background is not straightforward, but we have learned some key features: The algebraic special solutions of the Teukolsky equation will play a crucial role in finding the solution for the Hertz potential in terms of ψ_4 or ψ_0. One can see this as follows. In order to invert Eq. (2.7) for the potential we first seek out solutions of the homogeneous equation. Hence ψ_4 should vanish, but for the solution to be nontrivial, ψ_0, given by Eq. (2.6), must not vanish [22]. These two conditions are precisely the conditions that define the algebraically special solutions for the potential satisfying the vacuum Teukolsky equation (1.2). These two solutions (in the time domain) should allow one to build up the Kernel that inverts the fourth order equation (2.7). An identical argument applies for the ORG potential when working with Eqs. (2.8) and (2.9).

One main application of this formalism is to go beyond first order perturbation theory and compute second order perturbations of rotating black holes. In Ref. [16] there was developed a formalism for the second order Teukolsky equation that takes the form of the first order wave operator acting on the second order piece of the Weyl scalar ψ_4, and a *source* term build up as a quadratic combination of first order perturbations. It is precisely to compute this source term that one needs the explicit form of the metric perturbations. In Ref. [16] it was found that to describe the emitted gravitational radiation the ORG gauge is specially well suited. Hence one has to solve the the first order Teukolsky equation for $\psi_4^{(1)}$ and $\psi_0^{(1)}$, then later to build up the source of the second order piece of the Weyl scalar $\psi_4^{(2)}$ (See Eq. (9) in Ref. [16]).

A second important application of the reconstruction of metric perturbations around Kerr background is to compute the self force of a particle orbiting a massive black hole [23, 24] and to compute corrected trajectories [4] depending on the perturbed metric and connection coefficients along the particle world line. This task is left for a future paper. While we know the form of the Teukolsky equation in the presence of of perturbative matter around a Kerr hole (see Eq. (1.2)), we need to generalize the equation satisfied by the potential and the relationship between this potential and the metric perturbations, i.e. the generalization of Eqs. (2.3) and (2.5). In particular we know that not all of conditions (2.2) or (2.4) can hold in the presence of matter since they are then incompatible with the Einstein equations [25].

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APPENDIX A: RELATION BETWEEN IRG/ORG POTENTIALS

We give here a relation expressing the potential Ψ_{ORG} in terms of the result for Ψ_{IRG}. This was not given in [11, 12, 13, 14]. We begin by defining a field χ, with spin weight -2, through:

$$\Psi_{IRG} = \frac{1}{4}\left[\bar{\mathcal{L}}\bar{\mathcal{L}}\bar{\mathcal{L}}\bar{\mathcal{L}}\bar{\chi} + 12\rho^{-3}\psi_2\partial_t\chi\right], \tag{A1}$$

analogous to Eq. (2.9). The solution for Ψ_{ORG} is:

$$\Psi_{ORG} = DDDD\left[\bar{\chi}\right], \tag{A2}$$

also the relation (2.6) between ψ_0 and Ψ_{IRG}. A potential degeneracy for algebraically special modes was discussed briefly in [26]. The explicit solution for the modes $\chi^{\ell m}$ can be written, in terms of $\chi^{\pm}$ given by Eq. (3.4), as:

$$\begin{aligned}\chi^+(r) &= z_2(r)\int_{2M}^{r}\frac{y_2(r')V^+(r')}{\Delta(r')^2}\,dr' \\ &+ y_2(r)\int_{r}^{\infty}\frac{z_2(r')V^+(r')}{\Delta(r')^2}\,dr', \end{aligned}\tag{A3}$$

$$\begin{aligned}\chi^-(r) &= -y_1(r)\int_{2M}^{r}\frac{z_1(r')V^-(r')}{\Delta(r')^2}\,dr' \\ &- z_1(r)\int_{r}^{\infty}\frac{y_1(r')V^-(r')}{\Delta(r')^2}\,dr', \end{aligned}\tag{A4}$$

cf. Eqs. (3.14) and (3.15) above. Here $V^{\pm}$ are given by:

$$\begin{aligned}V^{\pm} &:= \frac{r^4}{3M\Delta}(\partial_t\Psi^{\pm}_{IRG}) \mp \frac{r^4}{3M\Delta}(\Omega_{AS})(\Psi^{\pm}_{IRG}) \\ &+4\frac{r}{3M}(Mr/\Delta - 1)(\Psi^{\pm}_{IRG}),\end{aligned}\tag{A5}$$

similar to Eq. (3.8), but note the sign differences.

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